Rješenje

1. (i) $f'(x_0) := \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

(ii) $f(x) := \sqrt{x}$,

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \quad \text{(proširimo racionalno)}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x})^2 - (\sqrt{x})^2}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

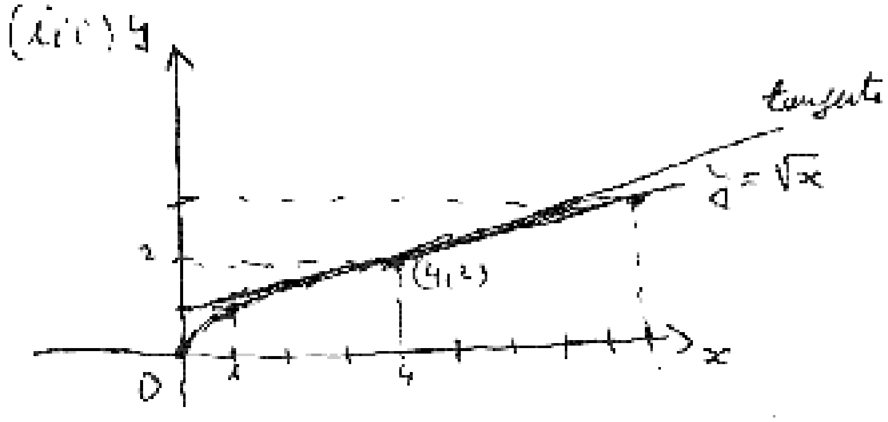
$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \quad \text{(kratimo } \Delta x\text{)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \quad \text{(uvrstimo } \Delta x = 0\text{)}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} .$$

(iii)



(iv) $y - f(x_0) = f'(x_0)(x - x_0)$

$f(x) := \sqrt{x}$; $x_0 = 4$

$f'(x) = \frac{1}{2\sqrt{x}}$; $f(x_0) = 2$

$f'(x_0) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

jednadžba tangente:

$$y - 2 = \frac{1}{4}(x - 4)$$

2. (i) $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x$

Može i: $f(x) \approx f(x_0) + f'(x_0)\cdot(x-x_0)$

(ii) kv. apr. $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2$

(može i: $f(x) \approx f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$)

kubna apr. $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \frac{f'''(x_0)}{3!}(\Delta x)^3$

(može i: $f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$)

(iii) $f(x) := e^x \quad x_0 = 0 \quad f(x_0) = e^0 = 1$

$f'(x) = e^x \quad f'(x_0) = 1$

$f''(x) = e^x \quad f''(x_0) = 1$

$f'''(x) = e^x \quad f'''(x_0) = 1$

lin. apr. $f(x) \approx 1 + x$

kv. apr. $f(x) \approx 1 + x + \frac{x^2}{2!}$

kubna apr. $f(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

(iv) $\sqrt{e} = e^{\frac{1}{2}}$ ~~pa ... $\sqrt{e} = f(\frac{1}{2})$ za~~

tj. $\sqrt{e} = f(\frac{1}{2})$ za $f(x) := e^x$.

lin. apr. $\sqrt{e} \approx 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

kv. apr. $\sqrt{e} \approx 1 + \frac{1}{2} + \frac{(\frac{1}{2})^2}{2!} = \frac{13}{8} = 1.625$

kubna apr. $\sqrt{e} \approx \frac{13}{8} + \frac{(\frac{1}{2})^3}{3!} = \frac{13}{8} + \frac{1}{48} = \frac{79}{48} \approx \ldots$

3. (i) $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

Može i ovako: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$

(ii) $f(x) := \frac{x^2-1}{x^2+1}$

$$f'(x) = \frac{(x^2-1)' \cdot (x^2+1) - (x^2-1) \cdot (x^2+1)'}{(x^2+1)^2}$$

$$= \frac{2x\,(x^2+1) - (x^2-1) \cdot 2x}{(x^2+1)^2} = \left(\frac{2x\left[(x^2+1) - (x^2-1)\right]}{(x^2+1)^2}\right)$$

$$= \frac{4x}{(x^2+1)^2} \quad //$$

(iii) $(g \circ f)' = (g' \circ f) \cdot f'$

Može i: $\underline{\underline{\left(g[f(x)]\right)' = g'[f(x)] \cdot f'(x)}}$

itd. (na primjer, da se zamijene f i g).

(iv) $f(x) := \sqrt{x^2+1}$ (Koristimo formulu za derivaciju složene funkcije)

$$f'(x) = \left(\sqrt{x^2+1}\right)'$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot (x^2+1)'$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+1}}$$

Potrebno je znati: $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

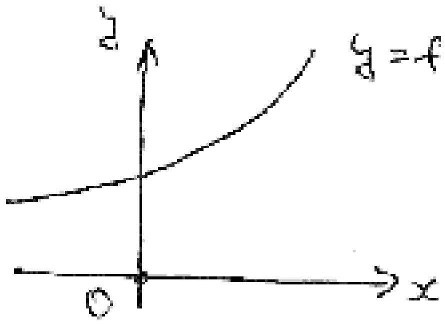
4\. (i) Ako je $f'(x_0)>0$ onda f raste oko x_0.

Ako je $f'(x_0)<0$ onda f pada oko x_0.

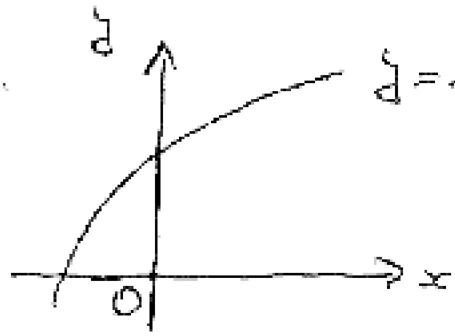
(ii) Ako je $f''(x_0)>0$ onda je f konveksna oko x_0

Ako je $f''(x_0)<0$ onda je f konkavna oko x_0.

(iii)

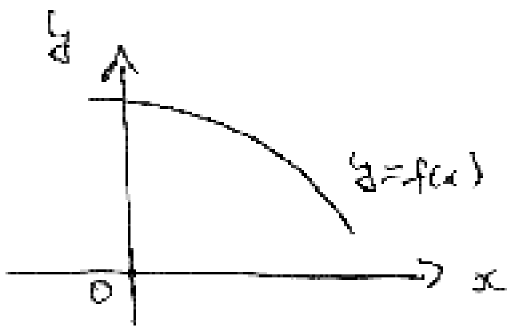


Ubrzani rast:
$f'(x)>0$ i $f''(x)>0$

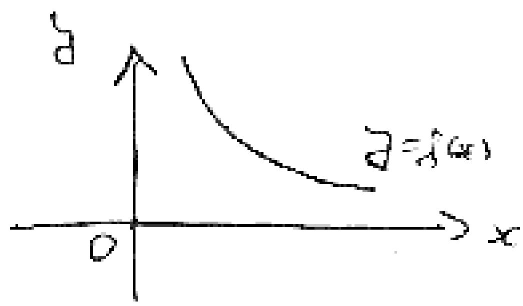


Usporeni rast:
$f'(x)>0$ i $f''(x)<0$

(iv)

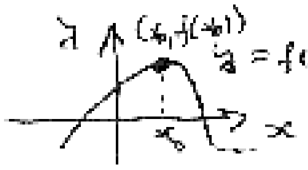


Ubrzani pad:
$f'(x)<0$ i $f''(x)<0$



Usporeni pad:
$f'(x)<0$ i $f''(x)>0$

5. (i) Funkcija f ima lokalni maksimum u x_0 ako je $f(x_0)$ najveće vrijednost funkcije oko x_0.

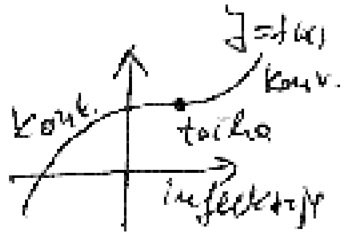


u x_0 je lokalni maksimum.

Funkcija f ima lokalni minimum u x_0 ako je $f(x_0)$ najmanje vrijednost funkcije f oko x_0.

u x_0 je lokalni minimum.

(ii) Točke infleksije su točke u kojima f prelazi iz konkavnosti u konveksnost ili obratno.



(iii) $f(x) := x^3 - 12x$; $f'(x) < 0$

$f'(x) = 3x^2 - 12$; $3x^2 - 12 < 0$

$f''(x) = 6x$; $x^2 < 4$

$-2 < x < 2$ (interval pada)

Zato:

$x < -2$ ili $x > 2$ (intervali rasta)

Zato je u $x_1 = -2$ lokalni maksimum, a u $x_2 = 2$ lokalni minimum.

$f''(x) > 0$

$6x > 0$

$x > 0$ (interval konv.)

$x < 0$ (interval konk.)

Zato je u $x_0 = 0$ točka infleksije.

(iv) Uz (iii) potrebno je još pronaći nultočke:

$x^3 - 12x = 0$

$x(x^2 - 12) = 0$

$x = 0$ ili $x = \pm\sqrt{12}$

$= \pm 2\sqrt{3}$

Treba preciznije od onoga